Defense Technical Information Center Compilation Part Notice

ADP014617

TITLE: Halt-Normal Plots for Multi-Level Factorial Experiments

DISTRIBUTION: Approved for public release, distribution unlimited

This paper is part of the following report:

TITLE: Proceedings of the Eighth Conference on the Design of Experiments in Army Research Development and Testing

To order the complete compilation report, use: ADA419759

The component part is provided here to allow users access to individually authored sections of proceedings, annals, symposia, etc. However, the component should be considered within the context of the overall compilation report and not as a stand-alone technical report.

The following component part numbers comprise the compilation report:
ADP014598 thru ADP014630

HALF-NORMAL PLOTS FOR MULTI-LEVEL FACTORIAL EXPERIMENTS

S. A. Krane
C-E-I-R, Inc., Dugway Field Office

1. INTRODUCTION. Half-normal plots for the interpretation of 2^p factorial experiments have been developed and popularized largely through the work of Cuthbert Daniel (see Daniel [1956] and [1959]). In this method the $2^p - 1$ main effects and interactions are estimated from observations on the 2^p treatment combinations. The empirical cumulative distribution of these estimates is then graphically compared with a cumulative distribution derived from a normal population. A rationale for this procedure is found in the approximate normality of the null distribution of the estimates, based upon normality of experimental errors or upon the tendency embodied in the Central Limit Theorem. According to Daniel, the half-normal plot permits the analyst to judge the reality of the largest main effects and interactions and serves to indicate bad values, heteroscedasticity, dependence of variance on mean and some types of defective randomization. The object of the present paper is to indicate and illustrate possible applications of half-normal plots to balance multi-level factorial experiments in general.

2. AN EXAMPLE. It appears easiest to introduce the technique of half-normal plotting for balanced multi-level factorial experiments in the context of a particular example. For this purpose we shall employ Example 8.1 of Davies [1954]. According to the authors (p. 291, "the data . . . are taken from the results of an investigation into the effects on the physical properties of vulcanized rubber of varying a number of factors, the property recorded being the wear resistance of the samples, and the factors being:

A five qualities of filler
B three methods of pretreatment of the rubber
C four qualities of the raw rubber . . ."

The data are reproduced in Table 1. From the data, the author develops the usual analysis of variance as shown in Table 2. The interpretation (Davies [1954, p. 296]) notes the significance of all main effects and two-factor interactions when tested against the three-factor interaction as error.

Table 1. (Table 8.1 of Davies [1954])

DATA OF A 5 x 3 x 4 FACTORIAL EXPERIMENT

WEAR RESISTANCE OF VULCANISED RUBBER

Level of factor A	Level of factor C											
	1			2			3			4		
	Level of factor B			Level of factor B			Level of factor B			Level of factor B		
	1	2	3	1	2	3	1	2	3	1	2	3
1	404	478	530	381	429	528	316	376	390	423	482	550
2	392	418	431	239	251	249	186	207	194	410	416	452
3	348	381	460	327	372	482	290	315	350	383	376	496
4	296	291	333	165	232	242	158	279	220	301	306	330
5	186	198	225	129	157	197	105	163	190	213	200	255

Table 2. (Table 8.16 of Davies [1954])

ANALYSIS OF VARIANCE OF TABLE 8.1

Source of Variation	Sum of squares	Degrees of freedom	Mean square	Variance ratio
Between levels of factor A ..	478,463	4	119,616	374+
B ..	52,794	2	26,397	82.5+
C ..	150,239	3	50,080	156+
Interactions AB	16,807	8	2,101	6.57+
AC	53,890	12	4,491	14.0+
BC	6,416	6	1,069	3.34*
Remainder = interaction ABC ..	7,688	24	320	
Total	766,297	59		

* Denotes significant, that is $\geq$ 5% value but < 1% value.

\+ Denotes highly significant, that is F $\geq$ 1% value.

In order to analyze the given experimental data by half-normal plotting, we shall reduce the data to single degree of freedom sums of squares. The method to be used depends upon the definition of complete sets of orthogonal contrasts for each of the factors A, B, and C. This definition generally is somewhat artibrary, but it is our experience that an experimenter familiar with the nature of the factor levels and the purpose of the experiment can, in most instances, provide sufficient justification for the prior definition of a meaningful complete set of single degree of freedom orthogonal contrasts among the levels. The use of orthogonal polynomials for quantitative levels is often indicated, while for qualitative levels, meaningful comparisons among certain levels are often obvious. On occasion, only a partial set of orthogonal comparisons will appear to be of intrinsic value and it may be necessary to complete the orthogonal set by adding contrasts of no apparent importance. In the absence of useful information on the nature of the levels (except that they are all qualitative) in the present example, we shall be totally arbitrary in defining the contrasts, but will attempt to indicate their potential interpretations. -These contrasts are shown in Table 3. For factor A, contrast A_0, is the "null" or "average" contrast*, while A_1 compares the average of levels 1 and 2 against the average of levels 3, 4 and 5, A_2 compares level 1 vs. level 2, A_3 compares level 3 against the average of levels 4 and 5 and A_4 compares level 4 with level 5. The orthogonality of the set is evident in that the coefficients sum to zero for all contrasts except the null contrast and the sum of products of coefficients is zero for all pairs of contrasts. For factor B, the non-null contrasts compare level 1 with level 2 with the average of levels 1 and 3. (In another context, B_1 and B_2 are the orthogonal polynomials for three equally spaced levels, B_1 being the linear contrast and B_2 the quadratic.) For factor C, the contrasts C_1, C_2 and C_3 make the following comparisons among levels, respectively: (1 and 2) vs. (3 and 4), (1 and 3) vs. (2 and 4) and (1 and 4)

---

* Daniel 1962 has suggested the term "null" is inappropriate because of the generally positive expectation of this contrast. We chose the term because (i) it is connoted by our zero subscript notation, (ii) this contrast is not a comparison among levels, and (iii) this contrast is generally "of no consequence" in the analysis.

Table 3. ORTHOGONAL CONTRASTS EMPLOYED

Factor A

Contrast	Level 1	2	3	4	5	Sum of Squares
A_0	+1	+1	+1	+1	+1	5
A_1	+3	+3	-2	-2	-2	30
A_2	+1	-1	0	0	0	2
A_3	0	0	+2	-1	-1	6
A_4	0	0	0	+1	-1	2

Factor B

Contrast	Level 1	2	3	Sum of Squares
B_0	+1	+1	+1	3
B_1	-1	0	+1	2
B_2	-1	+2	-1	6

Factor C

Contrast	Level 1	2	3	4	Sum of Squares
C_0	+1	+1	+1	+1	4
C_1	-1	-1	+1	+1	4
C_2	-1	+1	-1	+1	4
C_3	+1	-1	-1	+1	4

vs. (2 and 3). These contrasts would be of interest, e.g., in the event that factor C incorporated two subfactors, say D and E, where levels 1 and 2 are at the low level of D and levels 3 and 4 at the high level of D, while levels 1 and 3 are at the low level of E and 2 and 4 at the high level of E. Then C_1 is the effect of D, C_2 is the effect of E and C_3 is the interaction of D and E.

The three sets of contrasts (A_1, A_2, A_3, A_4), (B_1, B_2) and (C_1, C_2, C_3) will provide a basis for reducing the sums of squares for factor A (4 d.f.), factor B (2 d.f.) and factor C (3 d.f.) to independent single degree of freedom sums of squares. It remains to develop such a basis for the two- and three-factor interactions. A natural method for accomplishing this is the extension of the original single factor contrast sets to interaction contrast sets. This method is exemplified in Table 4 for Factors B and C. All possible combinations of the levels of B and C are employed as columns, while rows are contrasts. For any combination of a particular level, say i, of B with a particular level, say j, of C, the coefficient in the contrast B_qC_r is obtained by multiplication of the coefficient of level i of B in the contrast B_q by the coefficient of level j or C in the contrast C_r. Sums of squares of the B and C contrasts may be obtained by multiplication of the corresponding sums of squares for B and for C.

Of the 12 orthogonal contrasts in Table 4, B_0C_0 is the null contrast while the contrasts B_0C_1, B_0C_2, B_0C_3, B_1C_0 and B_2C_0 are simply the original contrasts C_1, C_2, C_3, B_1 and B_2, respectively, averaged over all levels of the other factor. The six contrasts B_1C_1, B_1C_2, B_1C_3, B_2C_1, B_2C_2, B_2C_3 are new and constitute a basis for partitioning the BC interaction sum of squares (6 d.f.) into orthogonal single degree of freedom sums of squares. Application of this method will likewise produce bases for partitioning the sums of squares for AB (8 d.f.), AC (12 d.f.) and ABC (24 d.f.).

The above method of defining interaction contrasts is incorporated in the method we employ for calculating half-normal variates by desk

Table 4. ORTHOGONAL CONTRASTS FOR FACTORS B AND C

	Level of C	1			2			3			4			Sum of
	Level of B	1	2	3	1	2	3	1	2	3	1	2	3	Squares
Contrast	B_0C_0	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	12
	B_0C_1	-1	-1	-1	-1	-1	-1	+1	+1	+1	+1	+1	+1	12
	B_0C_2	-1	-1	-1	+1	+1	+1	-1	-1	-1	+1	+1	+1	12
	B_0C_3	+1	+1	+1	-1	-1	-1	-1	-1	-1	+1	+1	+1	12
	B_1C_0	-1	0	+1	-1	0	+1	-1	0	+1	-1	0	+1	8
	B_1C_1	+1	0	-1	+1	0	-1	-1	0	+1	-1	0	+1	8
	B_1C_2	+1	0	-1	-1	0	+1	+1	0	-1	-1	0	+1	8
	B_1C_3	-1	0	+1	+1	0	-1	+1	0	-1	-1	0	+1	8
	B_2C_0	-1	+2	-1	-1	+2	-1	-1	+2	-1	-1	+2	-1	24
	B_2C_1	+1	-2	+1	+1	-2	+1	-1	+2	-1	-1	+2	-1	24
	B_2C_2	+1	-2	+1	-1	+2	-1	+1	-2	+1	-1	+2	-1	24
	B_2C_3	-1	+2	-1	+1	-2	+1	+1	-2	+1	-1	+2	-1	24

calculator. A sample work sheet for this method is shown in Table 5a and 5b*. Section I of this table is merely a recopying of the original data from Table 1. In this section a column of five numbers represents the observations for the five levels of A over a particular one of the twelve combinations of levels of B and C represented by columns. Section II is computed by operating on these columns with the contrasts A_0, A_1, A_2, A_3, A_4. For example, products of the coefficients of A_0 with the corresponding elements of a particular column are formed and these five products are summed and entered in Section II in the first row of that column. Similarly, sums of products of coefficients of A_1 with corresponding elements of columns are entered in the second row of Section II, and so forth. Thus each element of Section II is formed as a sum of products of coefficients of, say, A_p with corresponding observations and is entered on the (p+1)-th row of the appropriate column.

Section II may be visualized as an aggregation of 20 rows of three elements each, where each row corresponds to a particular contrast of A and a particular level of C. The three elements of each row correspond to the three levels of B. Now Section III is formed from Section II by summing products of coefficients of the B contrasts and corresponding elements of each row of three. Each such sum of products is entered in the corresponding row, with the sum of products from coefficients of B_q entered as the (q + 1)-th element of that row.

Section III may be visualized as comprising four sub-sections of 15 elements each, with the elements of a sub-section corresponding to a particular contrast of AB (i. e., a particular combination of a contrast of A and a contrast of B) and the four sub-sections corresponding to the four levels of C. Section IV is formed from Section III by summing products of coefficients of the C contrasts and identically placed elements from the four corresponding sub-sections. The sum of products is entered in the corresponding place of one of the sub-sections of Section IV, with the sum of products from coefficients of C_r entered in the (r+1)-th sub-section.

---

* This calculation method is essentially the same as that given in Appendix 8G of Davies [1954, pp. 363-6]. In some instances the format in Davies (with the addition of final column of half-normal variates) may be preferred.

Table 5a. COMPUTATION OF SINGLE DEGREES OF FREEDOM

	404	478	530	381	429	528	316	376	390	423	482	550
	392	418	431	239	251	249	186	207	194	410	416	452
I	348	381	460	327	372	482	290	315	350	383	376	496
	296	291	333	165	232	242	158	279	220	301	306	330
	186	198	225	129	157	197	105	163	190	213	200	255
	1626	1766	1979	1241	1441	1698	1055	1340	1344	1730	1780	2033
	728	948	847	618	518	489	400	235	232	705	930	844
II	12	60	99	142	178	279	130	169	196	13	66	98
	214	273	362	360	355	525	317	188	290	252	246	407
	110	93	108	36	75	45	53	116	30	88	106	75
	5371	353	-73	4380	457	-57	3739	289	281	5593	353	-253
	2523	119	321	1625	-129	-71	867	-168	-162	2479	139	-311
III	171	87	9	599	137	-65	495	66	12	177	85	21
	849	148	-30	1240	165	-175	795	-27	-231	905	155	-167
	311	-2	-32	156	9	69	199	-23	149	269	-13	49
	19083	1452	-102	-419	-168	158	863	168	-518	2845	-40	-550
	7494	-39	399	-802	-19	-101	714	59	81	2510	555	865
IV	1442	375	-23	-98	-73	89	110	69	-65	-746	-31	83
	3789	441	-603	-389	-185	-193	501	199	-81	-281	165	209
	935	-29	235	1	-43	161	-85	21	1	225	-1	-201

Table 5b. COMPUTATION ON SINGLE DEGREES OF FREEDOM (Cont'd)

V	$A_0B_0C_0$	$A_0B_1C_0$	$A_0B_2C_0$	$A_0B_0C_1$	$A_0B_1C_1$	$A_0B_2C_1$	$A_0B_0C_2$	$A_0B_1C_2$	$A_0B_2C_2$	$A_0B_0C_3$	$A_0B_1C_3$	$A_0B_2C_3$
	$A_1B_0C_0$	$A_1B_1C_0$	$A_1B_2C_0$	$A_1B_0C_1$	$A_1B_1C_1$	$A_1B_2C_1$	$A_1B_0C_2$	$A_1B_1C_2$	$A_1B_2C_2$	$A_1B_0C_3$	$A_1B_1C_3$	$A_1B_2C_3$
	$A_2B_0C_0$	$A_2B_1C_0$	$A_2B_2C_0$	$A_2B_0C_1$	$A_2B_1C_1$	$A_2B_2C_1$	$A_2B_0C_2$	$A_2B_1C_2$	$A_2B_2C_2$	$A_2B_0C_3$	$A_2B_1C_3$	$A_2B_2C_3$
	$A_3B_0C_0$	$A_3B_1C_0$	$A_3B_2C_0$	$A_3B_0C_1$	$A_3B_1C_1$	$A_3B_2C_1$	$A_3B_0C_2$	$A_3B_1C_2$	$A_3B_2C_2$	$A_3B_0C_3$	$A_3B_1C_3$	$A_3B_2C_3$
	$A_4B_0C_0$	$A_4B_1C_0$	$A_4B_2C_0$	$A_4B_0C_1$	$A_4B_1C_1$	$A_4B_2C_1$	$A_4B_0C_2$	$A_4B_1C_2$	$A_4B_2C_2$	$A_4B_0C_3$	$A_4B_1C_3$	$A_4B_2C_3$
VI	60	40	120	60	40	120	60	40	120	60	40	120
	360	240	720	360	240	720	360	240	720	360	240	720
	24	16	48	24	16	48	24	16	48	24	16	48
	72	48	144	72	48	144	72	48	144	72	48	144
	24	16	48	24	16	48	24	16	48	24	16	48
VII	6069348	52708	87	2926	706	208	12413	706	2236	134900	40	2521
	156000	6	221	1787	2	14	1416	15	9	17500	1283	1039
	86640	8789	11	400	333	165	504	298	88	23188	60	144
	199396	4052	2525	2102	713	259	3486	825	46	1097	567	303
	36426	53	1151	0	116	540	301	28	1	2109	0	842
VIII	2464	230	-9	-54	-27	14	111	27	-47	367	-6	-50
	395	-2	15	-42	-1	-4	38	4	3	132	36	32
	294	94	-3	-20	-18	13	22	17	-9	-152	-8	12
	447	64	-50	-46	-27	-16	59	29	-7	-33	24	17
	191	-7	34	0	-11	23	-17	5	1	46	-0	-29

The method of calculation of Sections II, III and IV may not be apparent at first glance, but verifying part or all of the data in Table 5a from the description above should help to clarify the process. Computing clerks will find it helpful to write the coefficients of each contrast on a strip of paper, appropriately oriented vertically or horizontally and spaced so that when overlaid on the worksheet each coefficient appears adjacent to the element to be multiplied.

Section V (Table 5b) merely identifies the elements of Section IV and subsequent sections according to the contrasts they represent. This identification is, of course, highly systematic and might well be omitted when familiarity with the method is attained.

Section VI contains the "divisors", obtained by multiplying the sums of squares of the coefficients of the contrasts A_p, B_q, C_r appropriate to each element, as found in Table 3.

Section VII contains the single degree of freedom sums of squares corresponding to each contrast. Each element is obtained by squaring an element of Section IV, dividing by the corresponding element of Section VI and entering in the corresponding place of Section VII.

Section VIII contains the half-normal variate values, each of which is computed as the square root of the corresponding element of Section VII, positive or negative according to the sign of the corresponding element of Section IV. (It would perhaps have been advisable to include the first decimal of each of these values in order to discriminate more fully among them.)

Certain check computations in the method have been omitted, but an over-all check can be readily obtained from Section VII by comparing sums of these single degree of freedom sums of squares with the usual analysis of variance of Table 2. These checks are indicated in Table 6. It will be noted that all sums of squares agree with Table 2 within the expected rounding error accumulated from Section VII.

The half-normal variates must now be ordered by magnitude before plotting. This ordering is shown in Table 7, along with an identification of the contrast represented (letters with subscripted zeroes have been dropped) and the appropriate quantile of the empirical distribution, defined by

Table 6. DEVELOPMENT OF USUAL ANALYSIS OF VARIANCE

Source	d.f.	Contrasts	S.S.
A	4	$A_1B_0C_0$ $A_2B_0C_0$ $A_3B_0C_0$ $A_4B_0C_0$	478462
B	2	$A_0B_1C_0$ $A_0B_2C_0$	52795
C	3	$A_0B_0C_1$ $A_0B_0C_2$ $A_0B_0C_3$	150239
AB	8	$A_1B_1C_0$ $A_2B_1C_0$ $A_3B_1C_0$ $A_4B_1C_0$	
		$A_1B_2C_0$ $A_2B_2C_0$ $A_3B_2C_0$ $A_4B_2C_0$	16808
AC	12	$A_1B_0C_1$ $A_2B_0C_1$ $A_3B_0C_1$ $A_4B_0C_1$	
		$A_1B_0C_2$ $A_2B_0C_2$ $A_3B_0C_2$ $A_4B_0C_2$	
		$A_1B_0C_3$ $A_2B_0C_3$ $A_3B_0C_3$ $A_4B_0C_3$	53890
BC	6	$A_0B_1C_1$ $A_0B_1C_2$ $A_0B_1C_3$	
		$A_0B_2C_1$ $A_0B_2C_2$ $A_0B_2C_3$	6417
ABC	24	$A_1B_1C_1$ $A_2B_1C_1$ $A_3B_1C_1$ $A_4B_1C_1$	
		$A_1B_1C_2$ $A_2B_1C_2$ $A_3B_1C_2$ $A_4B_1C_2$	
		$A_1B_1C_3$ $A_2B_1C_3$ $A_3B_1C_3$ $A_4B_1C_3$	
		$A_1B_2C_1$ $A_2B_2C_1$ $A_3B_2C_1$ $A_4B_2C_1$	
		$A_1B_2C_2$ $A_2B_2C_2$ $A_3B_2C_2$ $A_4B_2C_2$	
		$A_1B_2C_3$ $A_2B_2C_3$ $A_3B_2C_3$ $A_4B_2C_3$	7690
Total	59		766301
Mean	1	$A_0B_0C_0$	6069348
Raw total	60		6835649

Table 7. HALF-NORMAL VARIATES

Order k	Variate X_k	Contrast	Quantile P_k	Order k	Variate X_k	Contrast	Quantile P_k
60	2464	Null		30	24	$A_3B_1C_3$	.5000
59	447	A_3	.9915	29	23	$A_4B_1C_1$	.4831
58	395	A_1	.9746	28	22	A_2C_2	.4661
57	367	C_3	.9576	27	20	$-A_2C_1$	.4492
56	294	A_2	.9407	26	18	$-A_2B_1C_1$	.4322
55	230	B_1	.9237	25	17	$A_3B_2C_3$	.4153
54	191	A_4	.9068	24	17	$-A_4C_2$	.3983
53	152	$-A_2C_3$	.8898	23	17	$A_2B_1C_2$	.3814
52	132	A_1C_3	.8729	22	16	$-A_3B_2C_1$	.3644
51	111	C_2	.8559	21	15	A_1B_2	.3475
50	94	A_2B_1	.8390	20	14	B_2C_1	.3305
49	64	A_3B_1	.8220	19	13	$A_2B_2C_1$	.3136
48	59	A_3C_2	.8051	18	12	$A_2B_2C_3$	.2966
47	54	$-C_1$	.7881	17	11	$-A_4B_1C_1$	.2797
46	50	$-A_3B_2$	.7712	16	9	$-A_2B_2C_2$	.2627
45	50	$-B_2C_3$	.7542	15	9	$-B_2$	.2458
44	47	$-B_2C_2$	.7373	14	8	$-A_2B_1C_3$	.2288
43	46	A_4C_3	.7203	13	7	$-A_4B_1$	.2119
42	46	$-A_3C_1$	.7034	12	7	$-A_3B_2C_2$	.1949
41	42	$-A_1C_1$	.6864	11	6	$-B_1C_3$	.1780
40	38	A_1C_2	.6695	10	5	$A_4B_1C_2$	.1610
39	36	$A_1B_1C_3$	.6525	9	4	$A_1B_1C_2$	.1441
39	34	A_4B_2	.6356	8	4	$-A_1B_2C_1$	.1271
37	33	$-A_3C_3$	.6186	7	3	$-A_2B_2$	.1102
36	32	$A_1B_2C_3$	.6017	6	3	$A_1B_2C_2$	.0932
35	29	$-A_4B_2C_3$	.5847	5	2	$-A_1B_1$	.0763
34	29	$A_3B_1C_2$	.5678	4	1	$-A_1B_1C_1$	.0593
33	27	$-A_3B_1C_1$	.5508	3	1	$A_4B_2C_2$	.0424
32	27	$-B_1C_1$	.5339	2	0	$-A_4B_1C_3$	.0254
31	27	B_1C_2	.5169	1	0	A_4C_1	.0085

$$P_k = \frac{2k - 1}{2n}$$

where k is the rank order and n is the number of variates. Here, as in most instances, it seems appropriate that the null contrast be excluded from the variates to be examined. The sign of the contrast is now attached to the label and only positive variates are plotted.

The variate values and quantiles are next plotted on half-normal probability paper (as in Figure 1) for interpretation. Discussion of the interpretation phase of the analysis of this example will be deferred to a later section.

3. SOME THEORY*. At this point we shall touch briefly on some theoretical aspects of the development of half-normal variates from multi-level factorial experiments. To simplify the discussion we shall assume that we are concerned with a three-factor experiment, although it should be remembered that the theory and methodology apply with equal validity to any number of factors.

We denote by y_{hij} the observation obtained with factor A at level h, factor B at level i and factor C at level j, where $h = 1, 2, \ldots, a$; $i = 1, 2, \ldots, b$; $j = 1, 2, \ldots, c$. The coefficients of the orthogonal contrasts for factor A will be indicated by a_{ph}, denoting the coefficient for level h in the p-th contrast. Similarly the coefficients of the contrasts for factors B and C are denoted b_{qi} and c_{rj}, respectively.

We assume that for each factor there is a null contrast, these being denoted A_0, B_0, C_0 and defined by

$$a_{oh} = b_{oi} = c_{oj} = 1; \text{ all } h, i, j.$$

---

* This section is based on well-known results concerning distributions of linear functions of random variables and may be verified by reference to standard introductory texts on mathematical and theoretical statistics.

Furthermore, by the definition of orthogonal contrasts,

$$\sum_h a_{ph} = \sum_i b_{qi} = \sum_j c_{rj} = 0$$

$$p = 1, 2, \ldots, a-1; \quad q = 1, 2, \ldots, b-1; \quad r = 1, 2, \ldots, r-1;$$

and

$$\sum_h a_{ph} \; a_{p'h} = \sum_i b_{qi} \; b_{q'i} = \sum_j c_{rj} \, c_{r'j} = 0$$

$$p \neq p'; \quad q \neq q'; \quad r \neq r'.$$

The three-factor contrasts are defined by

$$(A_p B_q C_r) = \sum_h \sum_i \sum_j a_{ph} \, b_{qi} \, c_{rj} \, y_{hij};$$

$$p = 0, 1, \ldots, a-1; \quad q = 0, 1, \ldots, b-1; \quad r = 0, 1, \ldots, c-1.$$

Suppose that there are no treatment effects*, i.e.,

$$E\{y_{hij}\} = \mu; \text{ all } h, i, j;$$

and that the experimental errors are independent and have constant variance for all observations, i.e.,

$$E\{(y_{hij} - \mu)^2\} = \sigma^2; \text{ all } h, i, j.$$

---

*The symbol $E\{\}$ denotes the mathematical expectation operator.

Then

$$E\{A_p B_q C_r\} = 0,$$

unless

$$P = 0,\ q = 0 \text{ and } r = 0, \text{ in which case}$$

$$E\{A_0 B_0 C_0\} = abc\ \mu.$$

Furthermore*,

$$V\{A_p B_q C_r\} = (\sum_h a^2_{ph})(\sum_i b^2_{qi})(\sum_j c^2_{rj})\ \sigma^2.$$

Denote by Y_{pqr} the variate defined by

$$Y_{pqr} = (A_p B_q C_r) \Big/ \sqrt{(\sum_h a^2_{ph})\ (\sum_i b^2_{qi})\ (\sum_j c^2_{rj})}.$$

Then

$$E\{Y_{000}\} = \sqrt{abc}\ \mu;$$

$$E\{Y_{pqr}\} = 0, \text{ unless } p = 0,\ q = 0,\ r = 0;$$

$$V\{Y^2_{pqr}\} = \sigma^2.$$

If the experimental errors are normally distributed, then the Y_{pqr} are normally distributed. (Under fairly weak assumptions the Y_{pqr} will tend to be normally distributed in large experiments even for non-normal distributions of experimental error.) Then the non-negative half-normal variates,

---

*The symbol $V\{\ \}$ denotes the variance operator, $V\{X\} = E\{(X-E\{X\})^2\}$.

$$X_{pqr} = \left| Y_{pqr} \right| ,$$

$$= \sqrt{(A_p B_q C_r)^2 / (\sum_h a_{ph}^2)(\sum_i b_{qi}^2)(\sum_j c_{rj}^2)};\ p, q, r \neq 0, 0, 0;$$

are indeed distributed according to the half-normal density

$$f(x) = \sqrt{2/\pi\sigma^2} \quad \exp(-X^2/2\sigma^2), \qquad X \geq 0$$
$$X \leq 0.$$

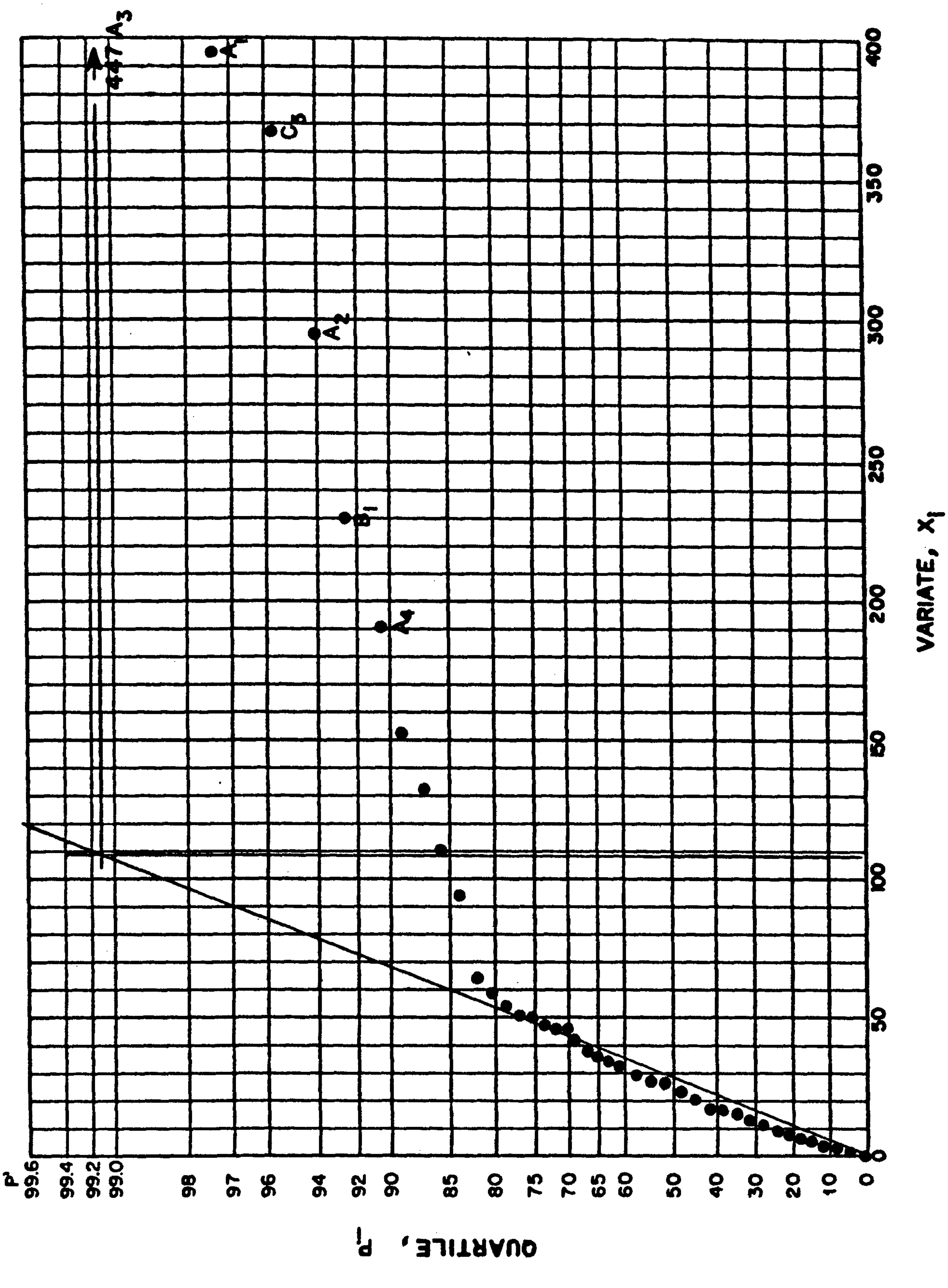
From this result, the half-normal variates for multi-level experiments may be seen to be essentially equivalent to those for 2^p experiments, making the work of Daniel 1959 and Birnbaum 1959 relevant to the interpretation.

4. INTERPRETATION OF EXAMPLE. We shall turn now to the interpretation of the example given earlier. Some difficulty will be experienced because of our ignorance of the precise nature of the factors and their levels, the experimental techniques and the observations themselves, but we shall attempt to proceed along lines suggested by Daniel for 2^p experiments.

To recapitulate the results of Section 2, we have, in Table 7, 59 ordered variates $X_k \equiv X_{pqr}$ whose empirical cumulative distribution should resemble the cumulative half-normal distribution under the hypothesis that there are no treatment effects. We have plotted these values against their quantiles in Figure 1, where they should be approximately linear under the null hypothesis.

We note at a glance that the plotted points are markedly and systematically non-linear. In fact, a little preliminary geometrical construction leads us to believe that a number of the variates are too large to have arisen by chance under the null hypothesis. The rationale for this belief is as follows. Under the null hypothesis the standard deviation, σ, is

FIGURE 1. HALF-NORMAL PLOT, ALL CONTRASTS



directly approximated by the value of X_m, where

$$m = (0.683\,n + 0.5),$$

$$= 41, \text{ approximately.}$$

From Table 7,

$$X_{41} = 42.$$

Then, under the null hypothesis, the plotted points should lie near a straight line through the origin and the point (X_m, P_m), indicated in Figure 1. Should the largest X lie "far enough" to the right of this line it is reasonable to presume that it did not arise by chance under the null hypothesis. It may then be taken as real and the next largest X promoted to the largest. This is roughly equivalent to increasing the ordinate of the second point to that of the first point. Should this replotted point also lie "far enough" to the right of the line, it too may be judged real and excluded, promoting the next X to the largest, etc. In Figure 1, we make a crude test of the largest values by constructing a horizontal through the largest point to intersect the previously constructed empirical cumulative distribution line. From this intersection we drop a vertical line and observe that all contrasts represented by points lying to the right of this vertical would have to be excluded before the largest X would lie on or above the original c. d. line. In this crude manner we judge from Figure 1 that six to ten of the largest values of X would be unlikely to occur under the null hypothesis. This graphical construction is no "exact" test; in fact it is rather likely that one or more contrasts would be judged "real" in this manner even if the null hypothesis did, in fact, hold. There is one element of conservativism in this procedure, in that the plotted c. d. line is based upon <u>all</u> contrasts, while a c. d. line based only on contrasts not judged "real" at this stage would lie to the left of the original line.

Let us tentatively suppose that the six largest contrasts (A_3, A_1, C_3, A_2, B_1, A_4) are real, considering (after Daniel [1959, p. 315]) their simple <u>names</u>, as well as their magnitudes relative to the rest of the set. We plot anew the 53 remaining contrasts in Figure 2. Actually, in addition to the ten largest remaining contrasts, only a fraction of the points are plotted, together with the c. d. line through (X_m, P_m), where

$$m = (0.683)(53) + 0.5,$$

$$= 37, \text{ approximately.}$$

The values of P_k are, of course, recalculated for $n = 53$. It appears reasonable to judge from this plot that the four largest contrasts (A_2C_3, A_1C_3, C_2, A_2B_1) are real.

A final plot of the values obtained after eliminating the ten largest values is shown in Figure 3. It appears in this plot that all real effects have been removed, with a residual error standard deviation approximately equal to

$$X^2_{34} = 841$$

(The actual mean square of the 49 residual contrasts is 816.)

Some further details of interpretation might be attempted. For example, there is a suggestion in Figure 1 and in Table 7 that there may have been plot-splitting, with factor B applied within plots. This also appears plausible from the rudimentary information given as to the nature of this factor. A further plotting, not shown here, in which contrasts including B_1 or B_2 were separated from those containing B_0 suggests a whole plot standard deviation of about 50-60 and a split-plot standard deviation of about 20-25.

5. COMPUTER USE. We have used half-normal plots for multi-level factorial experiments for almost two years. Our first major attempt to employ this technique was in the analysis of an unreplicated $10 \times 5 \times 3 \times 2^2$ experiment. The factor levels in this experiment were applied in a split-split-split plot design and certain problems of variance heterogeneity were apparent. The half-normal plotting of this data was sufficiently informative that it appeared worthwhile to develop a program for the IBM 1620 to be employed in computing half-normal variates from multi-level factorial data. This program, Single Degree of Freedom Analysis of Variance (SIDOF), has a capacity of eight factors, each at two to ten levels. It requires as input the observations and normalized vectors of contrast coefficients α_p, β_q, γ_r, etc., where

FIGURE 2. HALF-NORMAL PLOT, SIX LARGEST CONTRASTS OMITTED

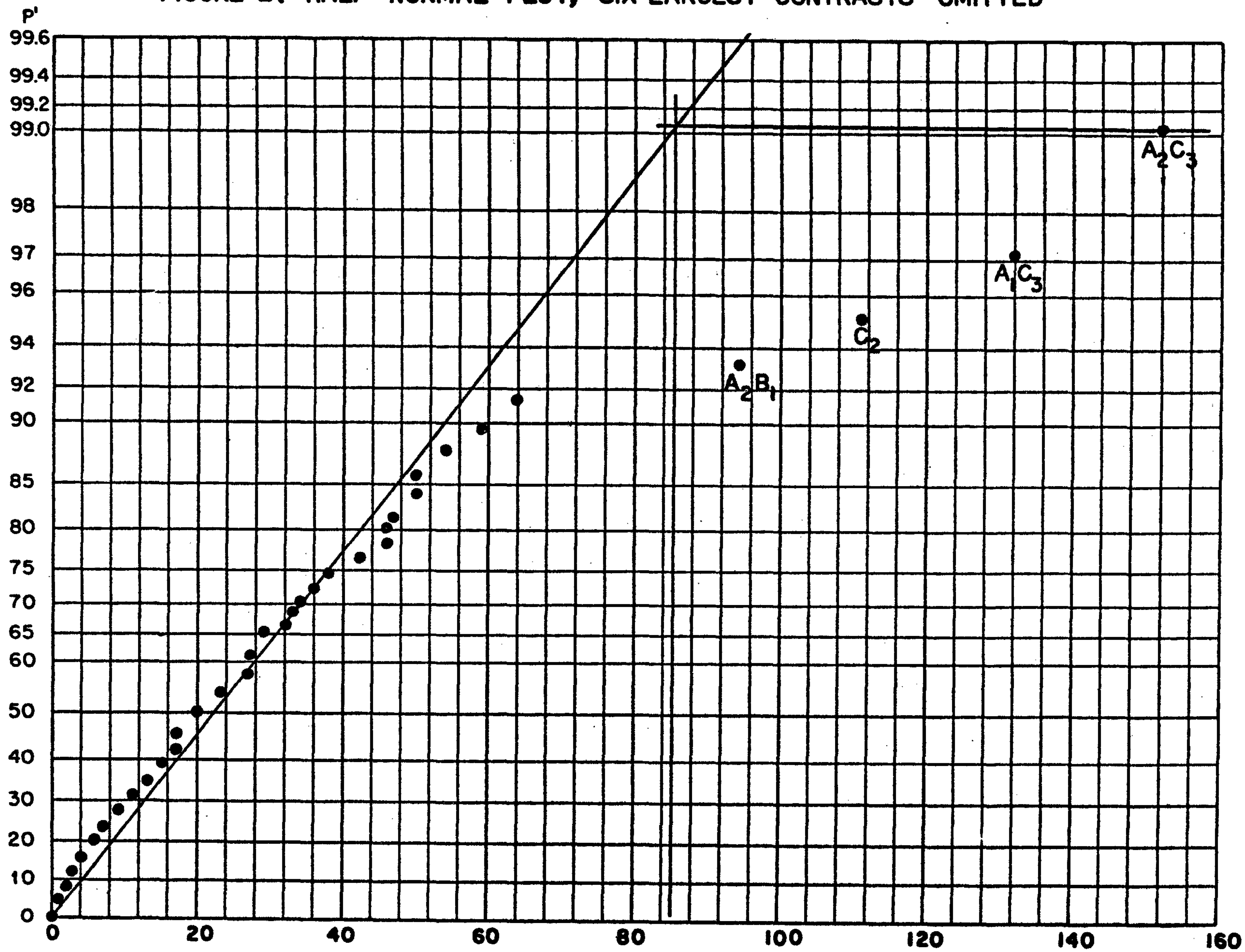
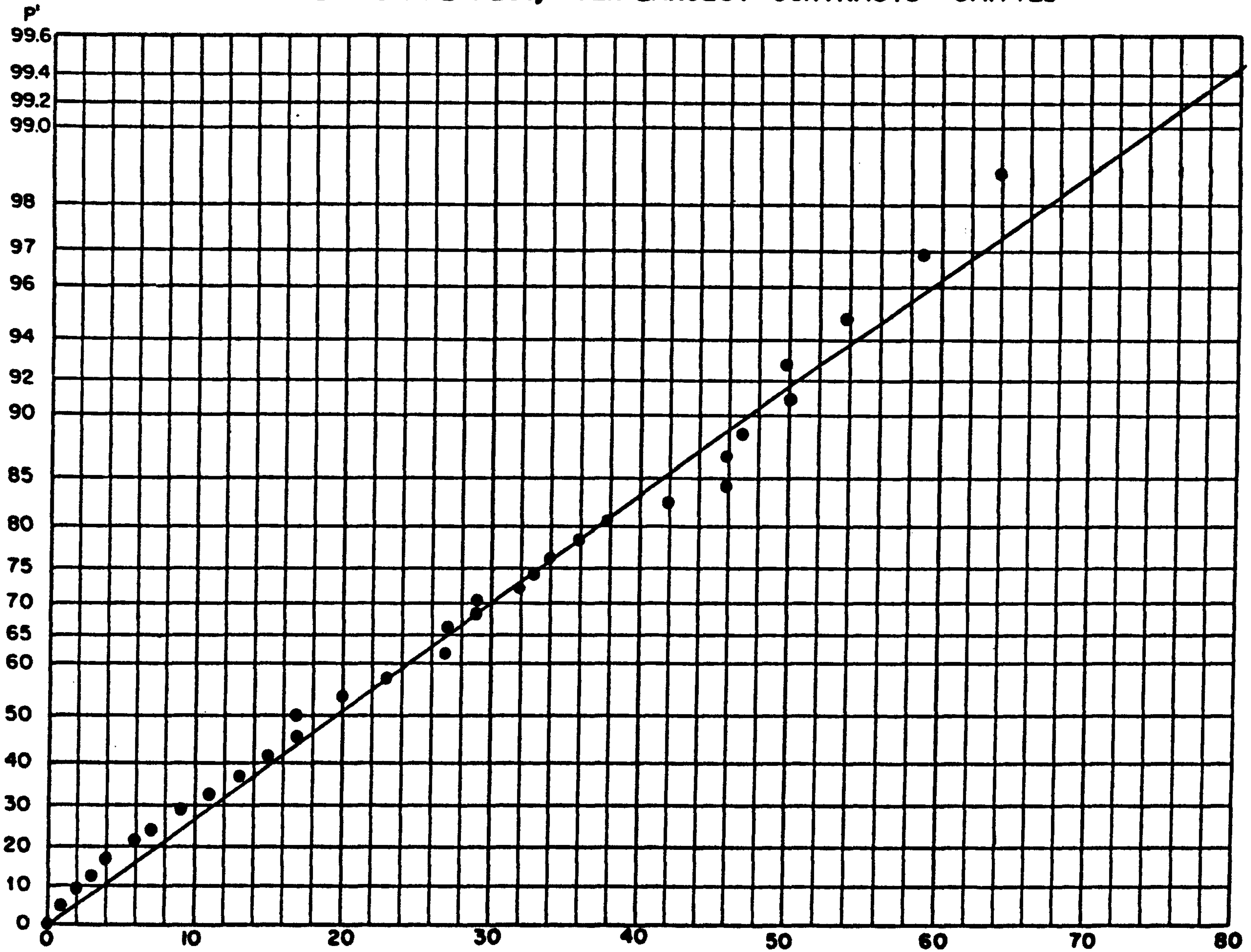


FIGURE 3. HALF-NORMAL PLOT, TEN LARGEST CONTRASTS OMITTED



$$\alpha_{ph} = a_{ph} / \sqrt{\sum_h a_{ph}^2}, \qquad h = 1, 2, \ldots, a;$$

$$\beta_{qi} = b_{qi} / \sqrt{\sum_i b_{qi}^2}, \qquad i = 1, 2, \ldots, b;$$

$$\gamma_{rj} = c_{rj} / \sqrt{\sum_j c_{rj}^2}, \qquad j = 1, 2, \ldots, c;$$

etc.

Each factor requires an additional "pass" through the machine. On the first pass, the machine computes the quantities (assuming three factors),

$$(A_p)_{ij} = \sum_h \alpha_{ph}\, y_{hij}.$$

On the second pass are computed the quantities.

$$(A_p B_q)_j = \sum_i \beta_{qi} (A_p)_{ij}$$

and on the third pass the quantities

$$(A_p B_q C_r) = \sum_j \gamma_{rj} (A_p B_q)_j.$$

At each pass the output includes both the (signed) contrasts developed and their squares. This program was one of the first developed for the IBM 1620 at Dugway Proving Ground and consequently was employed for a short period of time as a general-purpose analysis of variance. (It is, of course, much slower than other general-purpose programs available.)

6. EXPERIENCE. Some general comments on our experiences with half-normal plots for multi-level factorials may be in order. We shall be guided in this commentary largely by the approach of Daniel [1959].

a. Graph Sheets. We have generally used half-sheets of the Probability Scale x 90 Divisions paper available from Keuffel and Esser (Nos. 358-23 and 359-23).* Similar papers are available from several other sources. These papers are not particularly well-suited to the purpose. It would appear that special half-normal paper might be commercially feasible, but it is not, to our knowledge, currently available.

b. Birnbaum's test statistic. The test statistic developed by Allan Birnbaum [1959] has been used for our purposes. Birnbaum's work has been particularly oriented toward 2^p experiments and studies of the behavior of this statistic in multi-level factorials would be useful.

c. Defective values. Daniel indicates the utility of half-normal plotting in 2^p experiments for detecting defective values. For multi-level factorials the presence of defective values appears more difficult to diagnose, particularly with unrestricted sets of orthogonal contrasts. The isolation of the particular defective values is also more difficult.

d. Plot-splitting. The effect of plot-splitting upon the half-normal plots for multi-level experiments is similar to that described by Daniel. We have some reservations concerning indiscriminate searches for plot-splitting, however. It is generally accepted that in most experiments two-factor interactions tend to be smaller than main effects, three-factor interactions tend to be smaller than two-factor interactions, etc. (Here we are speaking of real effects and interactions, though perhaps of negligible magnitude.) Thus in actual experiments the slope of half-normal plots may be expected to increase with the relative number of high order interactions included. The plotted results of an experiment involving a number of small but real interactions may appear very similar to the results induced by plot-splitting, since split plot error contrasts invariably contain a relatively larger number of the higher order

*The graph sheets used in Figures 1, 2, and 3 were reproduced from a master kindly provided by Mr. Daniel. It is hoped that such sheets will soon be published.

contrasts. Our practice is generally to employ a split plot analysis only when knowledge of the experimental techniques indicates its propriety.

e. Convexity of plots. The detection of antilognormal distribution of error by downward convexity of half-normal plots appears difficult, as indicated by Daniel [1959, p. 336]. Most of our analysis work is, however, based on transformed data and we have seldom experienced this particular anomaly. In any event, the averaging effect of the contrasts would presumably minimize the effects of non-normality of error. On the other hand, we have noted that the removal of a moderate number of points representing apparently real effects often results in a downward convexity of the upper portion of the plot. We generally attribute this appearance to the inadvertent removal of one or more points representing error contrasts, for the result looks very much like the plot of a normal distribution with truncated upper tail.

7. REFERENCES.

Birnbaum, A. [1959], On the analysis of factorial experiments without replication, Technometrics I, No. 4, 343-57.

Davies, O. L., ed. [1954], The Design and Analysis of Industrial Experiments, Oliver and Boyd, London and Edinburgh.

Daniel, C. [1956], Fractional replication in industrial research, Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, Vol. V, 87-98, Univ. Of California Press, Berkeley and Los Angeles.

Daniel, C. [1959], Use of half-normal plots in interpreting factorial two-level experiments, Technometrics I, No. 4, 311-41.

Daniel, C. [1962], Personal Communication.